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A SUSY formulation à la Witten for the SUSY isotonic oscillator canonical supercoherent states

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Abstract. We extend Witten's supersymmetry (SUSY) formulation for Hamiltonian systems to a system of annihilation operator eigenvalue equations associated with the SUSY isotonic oscillator, which, as we show, define SUSY canonical supercoherent states containing mixtures of both pure bosonic and pure fermionic counterparts. Specifically, a graded Lie algebra structure analogous to Witten's SUSY quantum mechanical algebra is realized in which only annihilation operators participate, all expressed in terms of the Wigner annihilation operator of a related super-Wigner oscillator system.

1. Introduction

As is well known, the quantum mechanical (QM) $N = 2$ supersymmetry (SUSY) algebra of Witten [1, 2]

$$H_{ss} = \{Q_-, Q_+\} \qquad (Q_-)^2 = (Q_+)^2 = 0 \rightarrow [H_{ss}, Q_\mp] = 0 \tag{1}$$

involves† bosonic and fermionic sector Hamiltonians of the SUSY Hamiltonian H_{ss} (the even element), which get intertwined through the nilpotent charge operators $Q_- = (Q_+)^\dagger$ (the odd elements).

The extension of coherent states (CS) to super and SUSY systems has always attracted attention with applications for the usual QMSUSY oscillator as, for example, via the SUSY annihilation operator supercoherent states (SCS) definition by Aragone and Zypman [3] or the supergroup extension by Fatyaga *et al* [4] of the displacement operator definition of Perelomov CS [5] and studies by Balantekin, Schmitt and Barrett [6] for the supergroup $Osp(1/2N, \mathcal{R})$. However, the possibilities of a direct SUSY formulation à la Witten also for the SCS presumably associated with SUSY Hamiltonian systems (1) do not, to our knowledge, seem to have been reported so far. In this paper we show how this aim can be accomplished for the SUSY annihilation operator or canonical supercoherents states (CSCS) associated with a SUSY isotonic oscillator (harmonic plus a centripetal barrier) system.

† The anti-commutator of A and B is defined by $\{A, B\} \equiv AB + BA$, and the commutator of A and B is defined by $[A, B] \equiv AB - BA$.

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An influential work on representations and properties of a Wigner quantized [7] para-Bose oscillator CS is by Sharma, Mehta, Mukunda and Sudarshan [8] who, however, have concentrated their application to a purely Schrödinger description of these CS. Apart from the works in [8], parabosons of even and odd orders as irreducible constituents of the Fock space associated with Green's ansatz of two and three terms were also comprehensively studied by Macfarlane [9]. We extend the connection between the SUSY isotonic and super-Wigner oscillator Hamiltonian systems, pointed out recently by Jayaraman and Rodrigues [10], beyond the Hamiltonian level to the regime of the annihilation operators associated with these systems and show that a SUSY version for the isotonic CSCS can indeed be formulated.

The super-realization [10] of the Wigner-generalized oscillator has algebraic utilities for easier spectral resolutions of varied oscillator-related potentials [11]. It also has immediate potential applications for any full isotropic D-dimensional physical oscillator system [10] in its spherical version and has also attracted attention in quantum defect studies [12]. We therefore deem the present SUSY formulation of isotonic oscillator CSCS useful for possible extensions via the Wigner connection to SCS associated with other oscillator-related SUSY systems. Results of such investigations and the pursuit of current encouraging indications for extending the present formalism to SCS associated with other SUSY shape-invariant Hamiltonian systems [13] will be reported separately.

In the next section we realize a graded Lie algebra structure analogous to Witten's QMSUSY algebra for a system of annihilation operator eigenvalue equations associated with the SUSY isotonic oscillator. In section 3 we deduce the normalization, (non-)orthogonality and completeness relations for the SUSY CSCS and sharpen out the SUSY interwining relations between their bosonic and fermionic components. Finally, the constructions of position–momentum minimum uncertainty states as associated with the Wigner annihilation operator SCS and the SUSY annihilation operator SCS are briefly considered in section 4. In section 5 concluding remarks are given.

2. The graded Lie algebra structure

The connection between the SUSY isotonic Hamiltonian H_{ss} with unbroken SUSY and its charge operators $Q_{\mp}$ with the super-Wigner Hamiltonian $H(\ell+1) \equiv H$ and its ladder operators $a^-(\ell+1) \equiv a^- = (a^+)^\dagger$, $\ell+1 > 0$ being the Wigner parameter, is given by [10]:

$$H_{ss} = \begin{pmatrix} H_{ss}^- & 0 \\ 0 & H_{ss}^+ \end{pmatrix} = H - \tfrac{1}{2}\{\Sigma_3 + 2(\ell+1)\} \tag{2}$$

$$H_{ss}^- = H^-(\ell) - (\ell + \tfrac{3}{2}) \qquad H^+(\ell) \equiv H^-(\ell+1) \tag{3}$$

$$H = \begin{pmatrix} H^-(\ell) = \frac{1}{2}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{\ell(\ell+1)}{x^2}\right\} & 0 \\ 0 & H^+(\ell) \end{pmatrix} \tag{4}$$

$$Q_- = (Q_+)^\dagger = \tfrac{1}{2}(1-\Sigma_3)a^- = \Sigma_- A^- \tag{5}$$

$$a^- = \frac{1}{\sqrt{2}}\Sigma_1\left\{-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}(\ell+1)\Sigma_3 - x\right\} \tag{6}$$

$$A^- \equiv A^-(\ell+1) = \frac{1}{\sqrt{2}}\left\{-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}(\ell+1) - x\right\} = (A^+)^\dagger \tag{7}$$

$$(\Sigma_-)^2 = (\Sigma_+)^2 = 0 \qquad \{\Sigma_-, \Sigma_+\} = 1 \qquad \Sigma_-\Sigma_+ = N_f \tag{8}$$

where the fermionic coordinates are represented by $\Sigma_+ = \frac{1}{2}(\Sigma_1 + \mathrm{i}\Sigma_2)$ and $\Sigma_- = \frac{1}{2}(\Sigma_1 - \mathrm{i}\Sigma_2)$ with the usual Pauli matrix representation for $\vec{\Sigma}$ and with $N_f = \frac{1}{2}(1-\Sigma_3)$ being the fermion number operator. (We work with a natural system of units and take $\omega = 1$.)

While the QMSUSY algebra (1) is realized with the bosonic and fermionic sector Hamiltonians H_{ss}^- and H_{ss}^+ of H_{ss} of (2) given by

$$H_{ss}^- = A^+A^- \qquad H_{ss}^+ = A^-A^+ \tag{9}$$

$$Q_- = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} = (Q_+)^\dagger \tag{10}$$

the Wigner–Heisenberg (WH) algebra is realized by

$$H = \tfrac{1}{2}\{a^-, a^+\} \tag{11}$$

$$[H, a^-] = -a^- \qquad [H, a^+] = a^+ \tag{12}$$

$$[a^-, a^+] = 1 + 2(\ell + 1)\Sigma_3 \qquad \Sigma_3^2 = 1 \tag{13}$$

$$\{\Sigma_3, a^-\} = 0 \qquad \{\Sigma_3, a^+\} = 0 \to [\Sigma_3, H] = 0. \tag{14}$$

From (2), (4), (12) and (14) it follows that $A_{ss}^- = (a_-)^2$ is the SUSY annihilation operator, i.e. the annihilation operator for the unbroken SUSY spectrum of H_{ss}:

$$[H_{ss}, A_{ss}^-] = -2A_{ss}^- \qquad A_{ss}^- = (a^-)^2. \tag{15}$$

The isotonic SCS $|\alpha; \ell + 1\rangle_{ss} \equiv |\alpha\rangle_{ss}$, α being a complex number, are then defined here as the eigenstates of A_{ss}^- (following a convention as set in [3]):

$$A_{ss}^-|\alpha\rangle_{ss} = \alpha|\alpha\rangle_{ss} \tag{16}$$

$$A_{ss}^- = \begin{pmatrix} B^- \equiv B^-(\ell) & 0 \\ 0 & B^+ \equiv B^+(\ell) \end{pmatrix} \tag{17}$$

$$|\alpha\rangle_{ss} = \begin{pmatrix} |\alpha\rangle_- \equiv |\alpha; \ell\rangle_- \\ |\alpha\rangle_+ \equiv |\alpha; \ell\rangle_+ \end{pmatrix}. \tag{18}$$

That $|\alpha\rangle_{ss}$ are supersymmetric, true to the subscript notation employed, can now be easily proved.

On $\frac{1}{2}(1 + \Sigma_3)$ and $\frac{1}{2}(1 - \Sigma_3)$ projections, equations (16) and (15) decouple, in view of (2), (17) and (18), respectively, into

$$B^-|\alpha\rangle_- = \alpha_-|\alpha\rangle_- \qquad B^+|\alpha\rangle_+ = \alpha_+|\alpha\rangle_+ \tag{19}$$

$$[H_{ss}^-, B^-] = -2B^- \qquad [H_{ss}^+, B^+] = -2B^+ \tag{20}$$

where $B^\mp$ are given by

$$B^- = \tilde{A}^-A^- \tag{21}$$

$$B^+ = A^-\tilde{A}^- \qquad (B^+ = B^-(\ell + 1)) \tag{22}$$

$$B^- = \frac{1}{2}\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x} + x^2 - \frac{\ell(\ell + 1)}{x^2} + 1\right\} \tag{23}$$

$$\tilde{A}^- \equiv A^-\{-(\ell + 1)\} = \frac{1}{\sqrt{2}}\left\{-\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\ell + 1}{x} - x\right\} = (\tilde{A}^+)^\dagger. \tag{24}$$

Equations (19), (21) and (22) imply that $|\alpha\rangle_\mp$ get interwined in a SUSY-like manner:

$$|\alpha\rangle_- = C_-\tilde{A}^-|\alpha\rangle_+ \tag{25}$$

$$|\alpha\rangle_+ = C_+A^-|\alpha\rangle_- \tag{26}$$

$$C_- = \{_+ <\alpha|H_{ss}^+|\alpha>_+ + 2\ell + 1\}^{-\frac{1}{2}} \qquad C_+ = \{_- <\alpha|H_{ss}^-|\alpha>_-\}^{-\frac{1}{2}} \tag{27}$$

where the constants $C_\mp$ have been determined from normalizations on $|\alpha\rangle_\mp$ and the easily verifiable relation

$$\tilde{H}_{ss}^- = \tilde{A}^+\tilde{A}^- = H_{ss}^+ + 2\ell + 1 \qquad \tilde{H}_{ss}^+ = \tilde{A}^-\tilde{A}^+ = H_{ss}^- + 2\ell + 3 \tag{28}$$

together with (9). Note that $\tilde{H}_{ss}$ exhibits broken SUSY.

The graded Lie algebraic structure then readily follows with A^-_{ss} as the even element and with Q_- of (5) or (10) and

$$\tilde{Q}_- = \tfrac{1}{2}(1 + \Sigma_3)a^- = \begin{pmatrix} 0 & \tilde{A}^- \\ 0 & 0 \end{pmatrix} \tag{29}$$

as the odd elements:

$$\begin{aligned} &A^-_{ss} = \{Q_-, \tilde{Q}_-\} \qquad (Q_-)^2 = (\tilde{Q}_-)^2 = 0 \\ &\rightarrow [A^-_{ss}, Q_-] = [A^-_{ss}, \tilde{Q}_-] = 0 \end{aligned} \tag{30}$$

which is analogous to that of the QMSUSY algebra (equation (1)) of Witten for Hamiltonian systems but with the participation here only by annihilation operators, all expressed in terms of the super Wigner annihilation operator a^-. In terms of $a^- = Q_- + \tilde{Q}_-$ and $\mathrm{i}\Sigma_3 a^- = \mathrm{i}(\tilde{Q}_- - Q_-)$, an equivalent graded Lie algebraic structure to that in (30) also results:

$$A^-_{ss} = (a^-)^2 = (\mathrm{i}\Sigma_3 a^-)^2 \qquad \{a^-, \mathrm{i}\Sigma_3 a^-\} = 0 \tag{31}$$

which is analogous to the structure

$$H_{ss} = Q_1^2 = Q_2^2 \qquad \{Q_1, Q_2\} = 0 \tag{32}$$

equivalent to (1) with $Q_1 = Q_- + Q_+$ and $Q_2 = \mathrm{i}(Q_+ - Q_-)$.

Equations (15)–(24) also imply that the annihilation operators $B^\mp$ of respectively the bosonic and fermionic sector Hamiltonians $H^\mp_{ss}$ of H_{ss} are indeed the *SUSY partner* annihilation operators of the SUSY annihilation operator A^-_{ss} of H_{ss}, pertaining respectively to the bosonic ((*B*) with $N_f = 0$) and fermionic ((*F*) with $N_f = 1$) sectors of A^-_{ss} with their proper eigenstates $|\alpha\rangle_-$ and $|\alpha\rangle_+$, intertwined as in (25)–(27), being respectively the bosonic and fermionic components of $|\alpha\rangle_{ss}$:

$$|\alpha\rangle_{ss} = |\alpha\rangle_B + |\alpha\rangle_F \qquad |\alpha\rangle_B \equiv |\alpha; \ell + 1>_B \qquad |\alpha>_F \equiv |\alpha; \ell + 1>_F \tag{33}$$

$$|\alpha\rangle_B = \begin{pmatrix} |\alpha>_- \\ 0 \end{pmatrix} \qquad |\alpha\rangle_F = \begin{pmatrix} 0 \\ |\alpha\rangle_+ \end{pmatrix}. \tag{34}$$

Thus $|\alpha\rangle_{ss}$ are supersymmetric CSCS containing mixtures of purely bosonic $|\alpha\rangle_B$ and purely fermionic $|\alpha\rangle_F$ counterparts.

3. The completeness of SUSY CSCS

The explicit expressions for $|\alpha\rangle_-$ and $|\alpha\rangle_+$,

$$|\alpha\rangle_- = \sum_{m=0}^{\infty} b^-_m |m\rangle_- \qquad |\alpha\rangle_+ = \sum_{m=0}^{\infty} b^+_m |m\rangle_+ \tag{35}$$

$$H^-_{ss}|m\rangle_- = \varepsilon^{(m)}_{ss;-}|m\rangle_- \qquad H^+_{ss}|m\rangle_+ = \varepsilon^{(m)}_{ss;+}|m\rangle_+ \tag{36}$$

in terms of the complete orthonormal set of eigenstates $|m\rangle_-(\equiv |\tilde{m}\rangle_+)$ and $|m\rangle_+(\equiv |\tilde{m}\rangle_-)$ of $H^-_{ss}(\tilde{H}^+_{ss})$ and $H^+_{ss}(\tilde{H}^-_{ss})$ for the energies $\varepsilon^{(m)}_{ss;-}(\tilde{\varepsilon}^{(m)}_{ss;+})$ and $\varepsilon^{(m)}_{ss;+}(\tilde{\varepsilon}^{(m)}_{ss;-})$ can be determined with the use of the SUSY interwining relations:

$$|m+1\rangle_- = \frac{1}{\sqrt{\varepsilon^{(m)}_{ss;+}}} A^+|m\rangle_+ \qquad |m\rangle_+ = \frac{1}{\sqrt{\varepsilon^{(m+1)}_{ss;-}}} A^-|m+1\rangle_- \tag{37}$$

$$|\tilde{m}\rangle_- = \frac{1}{\sqrt{\tilde{\varepsilon}^{(m)}_{ss;+}}} \tilde{A}^+|\tilde{m}\rangle_+ \qquad |\tilde{m}\rangle_+ = \frac{1}{\sqrt{\tilde{\varepsilon}^{(m)}_{ss;-}}} \tilde{A}^-|\tilde{m}\rangle_- \tag{38}$$

$$\begin{aligned}&\varepsilon_{ss;-}^{(m)} \equiv \varepsilon_{ss;-}^{(m)}(\ell) = 2m \qquad \varepsilon_{ss;+}^{(m)} \equiv \varepsilon_{ss;+}^{(m)}(\ell) = \varepsilon_{ss;-}^{(m+1)}\\ &\tilde{\varepsilon}_{ss;-}^{(m)} = \varepsilon_{ss;+}^{(m)} + 2\ell + 1 = \tilde{\varepsilon}_{ss;+}^{(m)}\\ &\tilde{\varepsilon}_{ss;+}^{(m)} = \varepsilon_{ss;-}^{(m)} + 2\ell + 3 = \tilde{\varepsilon}_{ss;-}^{(m)} \qquad (m = 0, 1, \ldots).\end{aligned} \tag{39}$$

First, from (21), (22) and (37)–(39) one derives that

$$B^{\mp}|m\rangle_{\mp} = \sqrt{2m(2m + 2\ell + 2 \mp 1)}|m - 1\rangle_{\mp} \tag{40}$$

$$(B^{\mp})^{\dagger}|m\rangle_{\mp} = \sqrt{2(m + 1)(2m + 2\ell + 4 \mp 1)}|m + 1\rangle_{\mp} \tag{41}$$

and hence that

$$|m\rangle_{\mp} = \left\{\frac{\Gamma(\ell + 2 \mp \frac{1}{2})}{2^{2m} m!\Gamma(\ell + 2 \mp \frac{1}{2} + m)}\right\}^{\frac{1}{2}} [(B^{\mp})^{\dagger}]^m |0\rangle_{\mp}. \tag{42}$$

Insertion then of (40) and (35) into (19) leads to the recursion relation

$$b_m^{\mp} = \frac{\alpha}{2}\left\{m\left(m + \ell + 1 \mp \frac{1}{2}\right)\right\}^{-\frac{1}{2}} b_{m-1}^{\mp} \tag{43}$$

and thereby to a straightforward determination of the normalized $|\alpha\rangle_{\mp}$:

$$|\alpha\rangle_{\mp} = (g_{\mp})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{1}{\{m!\Gamma(m + \ell + 2 \mp \frac{1}{2})\}^{\frac{1}{2}}} \left(\frac{\alpha}{2}\right)^m |m\rangle_{\mp} \tag{44}$$

$$g_{\mp} \equiv g_{\mp}(|\alpha|) = \left(\frac{2}{|\alpha|}\right)^{\ell+1\mp\frac{1}{2}} I_{\ell+1\mp\frac{1}{2}}(|\alpha|) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m + \ell + 2 \mp \frac{1}{2})}\left(\frac{|\alpha|}{2}\right)^{2m} \tag{45}$$

$$= \Gamma(\ell + 2 \mp \tfrac{1}{2})^{-1}\, {}_0F_1\left(\ell + 2 \mp \frac{1}{2}; \frac{|\alpha|^2}{4}\right) \tag{46}$$

$${}_{\mp}\langle\xi; \ell|\alpha; \ell\rangle_{\mp} = \{g_{\mp}(|\alpha|)g_{\mp}(|\xi|)\}^{-\frac{1}{2}} g_{\mp}[(\xi^*\alpha)^{\frac{1}{2}}]. \tag{47}$$

In the above, Γ represents the familiar Gamma function, I_k denotes the first modified Bessel function of the kth order [14] and ${}_0F_1$ is the doubly confluent hypergeometric function. Equation (47) expresses the non-orthogonality relation for $|\alpha\rangle_{\mp}$.

By virtue of (42) and (46), the expression (44) for $|\alpha\rangle_{\mp}$ can be put in the form

$$|\alpha\rangle_{\mp} = \left\{{}_0F_1\left(\ell + 2 \mp \frac{1}{2}; \frac{|\alpha|^2}{4}\right)\right\}^{-\frac{1}{2}} {}_0F_1\left(\ell + 2 \mp \frac{1}{2}; \alpha\frac{(B^{\mp})^{\dagger}}{4}\right)|0\rangle_{\mp} \tag{48}$$

which is reminiscent of the *displacement operator* definition for the usual oscillator coherent states. Note in (48) that $|0 >_{\mp} = |m = 0\rangle_{\mp} = |\alpha = 0\rangle_{\mp}$ which means that the ground states are members of the sets of coherent states obtained via equation (48).

The procedure for determining the positive definite integration measure $\bar{\mu}_{\mp}(\alpha)$ entering into the completeness relation

$$\frac{1}{2\pi}\int |\alpha\rangle_{\mp\,\mp}\langle\alpha|\bar{\mu}_{\mp}(\alpha)\,\mathrm{d}^2\alpha = \sum_{m=0}^{\infty} |m\rangle_{\mp\,\mp}\langle m| = 1 \tag{49}$$

leads to a moment problem [8,15] already familiar in the theory of para-Bose oscillator CS [8]:

$$\int_0^{\infty} g_{\mp}^{-1}\mu_{\mp}(|\alpha|)(|\alpha|)^{2m+1}\,\mathrm{d}|\alpha| = 2^{2m} m!\Gamma(m + \ell + 2 \mp \tfrac{1}{2}) \qquad \bar{\mu}_{\mp}(\alpha) = \mu_{\mp}(|\alpha|) \tag{50}$$

and can be solved [8] by invoking the equality

$$\int_0^{\infty} K_{\nu}(|\alpha|)(|\alpha|)^{\lambda}\,\mathrm{d}|\alpha| = 2^{\lambda-1}\Gamma\left(\frac{\lambda + \nu + 1}{2}\right)\Gamma\left(\frac{\lambda - \nu + 1}{2}\right) \tag{51}$$

(Gradshteyn and Ryzhik [16]). In fact, with the correspondence of $\lambda = 2m + \ell + 2 \mp \frac{1}{2}$ and $\nu = \ell + 1 \mp \frac{1}{2}$ one obtains that

$$\bar{\mu}_{\mp}(\alpha) = \mu_{\mp}(|\alpha|) \equiv \mu_{\mp} = I_{\ell+1\mp\frac{1}{2}}(|\alpha|)K_{\ell+1\mp\frac{1}{2}}(|\alpha|) \tag{52}$$

where K_k denotes the second modified Bessel function of the kth order [14].

The states $|\alpha\rangle_{\mp}$ are also incidentally the ordinary CCS associated with the individual isotonic oscillator Hamiltonians $H^{\mp}(\ell)$ of (4). Thus our SUSY fomulation of isotonic SCS has yielded their constructions in a SUSY way, satisfying normalization, non-orthogonality and completeness relations as derived here. The completeness property of the SUSY CSCS $|\alpha\rangle_{ss}$ of (33) and (34) results then from that of its bosonic and fermionic components in equations (49) and (52):

$$I(2\times 2) = \frac{1}{2\pi}\int \{|\alpha\rangle_{B\ B}\langle\alpha|\mu_- + |\alpha\rangle_{F\ F}\langle\alpha|\mu_+\}\, \mathrm{d}^2\alpha \tag{53}$$

$$= \frac{1}{2\pi}\int \{\tfrac{1}{2}(1+\Sigma_3)|\alpha\rangle_{ss\ ss}\langle\alpha|\tfrac{1}{2}(1+\Sigma_3)\mu_- + \tfrac{1}{2}(1-\Sigma_3)|\alpha\rangle_{ss\ ss}\langle\alpha|\tfrac{1}{2}(1-\Sigma_3)\mu_+\}\, \mathrm{d}^2\alpha \tag{54}$$

$$_F\langle\xi;\ell+1|\alpha;\ell+1\rangle_B = 0. \tag{55}$$

Also, by virtue of the forms (10) and (29) for Q_- and $\tilde{Q}_-$, the graded structure (30), the SUSY intertwining Hamiltonian eigenstates relations (37)–(39) and the explicit expressions derived for $|\alpha\rangle_{\mp}$ in (44)–(46), the SUSY intertwining relations (25)–(27) for the bosonic and fermionic components $|\alpha\rangle_{\mp}$ can be sharpened into the following suggestive forms:

$$Q_-|\alpha\rangle_B = \frac{\alpha}{\sqrt{2}}\left\{\frac{g_+}{g_-}\right\}^{\frac{1}{2}}|\alpha\rangle_F \qquad \tilde{Q}_-|\alpha\rangle_F = \sqrt{2}\left\{\frac{g_-}{g_+}\right\}^{\frac{1}{2}}|\alpha\rangle_B. \tag{56}$$

Relations in (56) together with the use of (10) and (29) for Q_- and $\tilde{Q}_-$ also lead to the normalized eigenstates $|+z;\ell+1\rangle_{\mathrm{W}} \equiv |+z\rangle_{\mathrm{W}}$, $|-z;\ell+1\rangle_{\mathrm{W}} \equiv |-z\rangle_{\mathrm{W}}$, $z = \sqrt{\alpha}$, of the super Wigner annihilation operator,

$$a|\pm z\rangle_{\mathrm{W}} = \pm z|\pm z\rangle_{\mathrm{W}} \qquad {}_{\mathrm{W}}\langle\pm z|\pm z\rangle_{\mathrm{W}} = 1 \tag{57}$$

which obtain their explicit forms as the special superpositions:

$$|\pm z\rangle_{\mathrm{W}} = \frac{1}{(g_- + \frac{|\alpha|}{2}g_+)^{\frac{1}{2}}}\left\{(g_-)^{\frac{1}{2}}|\alpha\rangle_B \pm \frac{z}{\sqrt{2}}(g_+)^{\frac{1}{2}}|\alpha\rangle_F\right\} \qquad \sqrt{\alpha} = z. \tag{58}$$

The completeness property of $|+z\rangle_{\mathrm{W}}$ and $|-z\rangle_{\mathrm{W}}$ follows from that for the SUSY CSCS $|\alpha\rangle_{ss}$ (equations (53), (54)), and the resolution of this identity relation easily expressible in terms of $|-z\rangle_{\mathrm{W}}$ and $|+z\rangle_{\mathrm{W}}$ through the use of (58) will then contain, as well as the diagonal entries $|z\rangle_{\mathrm{W}\ \mathrm{W}}\langle z|$, off-diagonal entries $|z\rangle_{\mathrm{W}\ \mathrm{W}}\langle -z|$ for the non-vanishing value, as assumed here, of the Wigner parameter $\ell + 1$:

$$\frac{1}{\pi 2^{\ell+\frac{1}{2}}}\int\{|z\rangle_{\mathrm{W}_{(+)}\ \mathrm{W}_{(+)}}\langle z|K_{\ell+\frac{1}{2}} + |z\rangle_{\mathrm{W}_{(-)}\ \mathrm{W}_{(-)}}\langle z|K_{\ell+\frac{3}{2}}\}\left\{g_- + \frac{|z|^2}{2}g_+\right\}|z|^{2\ell+3}\,\mathrm{d}^2z = I(2\times 2)$$
$$K_k \equiv K_k(|z|^2) \tag{59}$$

$$|z\rangle_{\mathrm{W}_{(+)}} = \tfrac{1}{2}(|z\rangle_{\mathrm{W}} + |-z\rangle_{\mathrm{W}}) \qquad |z\rangle_{\mathrm{W}_{(-)}} = \tfrac{1}{2}(|z\rangle_{\mathrm{W}} - |-z\rangle_{\mathrm{W}}). \tag{60}$$

The fact that when z covers the complex plane once, α covers it twice has been accounted for in the above completeness relation. The feature of off-diagonal entries in the identity operator (59) governed here by a SUSY formulation of CSCS corresponds, in fact, to the SUSY version of a similar feature for para-Bose CS first discussed by Sharma *et al* [8], applied by them to a purely Schrödinger description of CS. It may be noted that $|\pm z\rangle_{\mathrm{W}}$ are not eigenstates of $\mathrm{i}\Sigma_3 a^-$ owing to the graded structure (31):

$$\begin{aligned} \mathrm{i}\Sigma_3 a^-|+z\rangle_{\mathrm{W}} &= \mathrm{i}z\Sigma_3|+z\rangle_{\mathrm{W}} = \mathrm{i}z|-z\rangle_{\mathrm{W}} \\ \mathrm{i}\Sigma_3 a^-|-z\rangle_{\mathrm{W}} &= -\mathrm{i}z\Sigma_3|-z\rangle_{\mathrm{W}} = -\mathrm{i}z|+z\rangle_{\mathrm{W}}. \end{aligned} \tag{61}$$

4. The minimum uncertainty SCS

To complete our analysis, we trace below the construction of minimum uncertainty coherent states (MUCS) for the Wigner position $\hat{x}$ and momentum $\hat{p}$ defined by

$$\sqrt{2}a^{\mp} = (\mp\mathrm{i}\hat{p} - \hat{x}) \tag{62}$$

$$\rightarrow \hat{x} = \Sigma_1 x \qquad \hat{p} = \Sigma_1\left\{-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{i}}{x}(\ell+1)\Sigma_3\right\} \tag{63}$$

which satisfy the super generalized quantum commutation relation

$$[\hat{x}, \hat{p}] = \mathrm{i}\{1 + 2(\ell+1)\Sigma_3\} \tag{64}$$

by virtue of (13). Developing the usual procedure for the construction of minimum uncertainty states $|\psi\rangle_M$ with equal dispersions for position and momentum, one has

$$a^-|\psi\rangle_M = z|\psi\rangle_M \qquad z = \sqrt{\alpha} = -\frac{1}{\sqrt{2}}(\langle\hat{x}\rangle + \mathrm{i}\langle\hat{p}\rangle) \tag{65}$$

which identifies, in view of (57) and (58), that $|\psi\rangle_M = |z\rangle_{\mathrm{W}}$ are a particular set of SUSY CSCS, being eigenstates of A^-_{ss} and as well a^-.

The entire class of SUSY CSCS defined by (16) can still be associated with minimum uncertainty SCS but only with new definitions of the position $\hat{X}$ and momentum $\hat{P}$ whose expressions then stem *naturally* from those for the SUSY annihilation and creation operators $A^-_{ss}(=\{A^+_{ss}\}^\dagger)$ analogously to the way that $\hat{x}$ and $\hat{p}$ were defined in (62):

$$A^{\mp}_{ss} = \frac{1}{\sqrt{2}}(\mp\mathrm{i}\hat{P} - \hat{X}) \tag{66}$$

$$\rightarrow \hat{X} = 2\sqrt{2}(x^2 - H_{ss}) \qquad \hat{P} = \mathrm{i}\sqrt{2}\left(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\right) \tag{67}$$

$$\rightarrow [\hat{X}, \hat{P}] = 4\mathrm{i}H = 4\mathrm{i}\{H_{ss} + \tfrac{1}{2}[\Sigma_3 + 2(\ell+1)]\}. \tag{68}$$

Then, the procedure for the construction of the minimum uncertainty states $|\Psi\rangle_M$ with equal dispersions for $\hat{X}$ and $\hat{P}$ leads to

$$A^-_{ss}|\Psi\rangle_M = \alpha|\Psi\rangle_M \qquad \alpha = -\frac{1}{\sqrt{2}}(\langle\hat{X}\rangle + \mathrm{i}\langle\hat{P}\rangle) \tag{69}$$

which identifies, in view of (16), that $|\Psi\rangle_M = |\alpha\rangle_{ss}$ are indeed the SUSY CSCS. It may be observed that the $\hat{X}$ and $\hat{P}$ operators obtained in the SUSY way in (66)–(68) coincide with the *natural* quantum operators used by Nieto [17] on projections respectively to the bosonic and fermionic sectors of $A^{\mp}_{ss}$. For this reason, the bosonic and fermionic components $|\alpha >_{\mp}$, equations (44)–(48), also coincide with the MUCS of Nieto [17] employing his *natural* classical to quantum variables method for the oscillator with centripetal barrier.

In figures 1 and 2 we plot the minimum uncertainty CSCS $(\Delta\hat{X})(\Delta\hat{P})$ and the expectation value $\langle\hat{X}^2\rangle$ as a function of α for $\ell = 1$. However, the minimum uncertainty $(\Delta\hat{x})(\Delta\hat{p})$ for the Wigner oscillator CS, we find, is a constant.

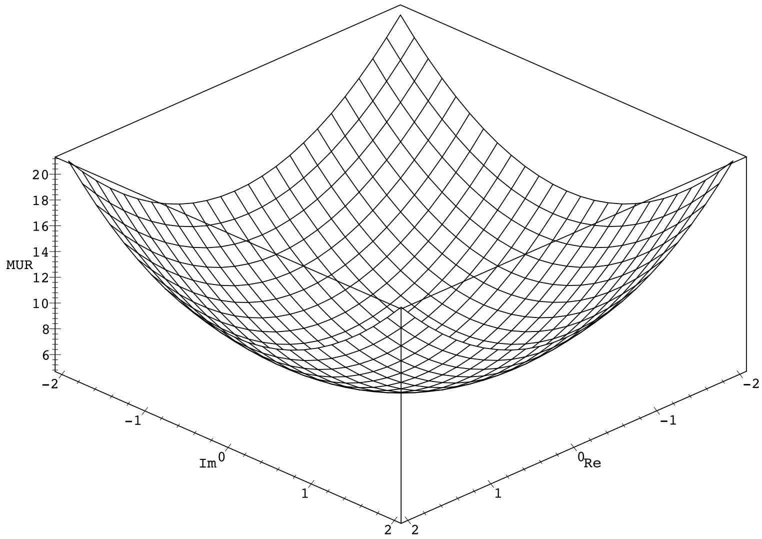

Figure 1. The minimum uncertainty $(\Delta\hat{X})(\Delta\hat{P})$ of CSCS, as a function of α, for the particular case of $\ell = 1$. In this figure $MUR \equiv \Delta\hat{X}\Delta\hat{P}$, Re $\equiv \mathrm{Re}\,(\alpha)$ and Im $\equiv \mathrm{Im}\,(\alpha)$.

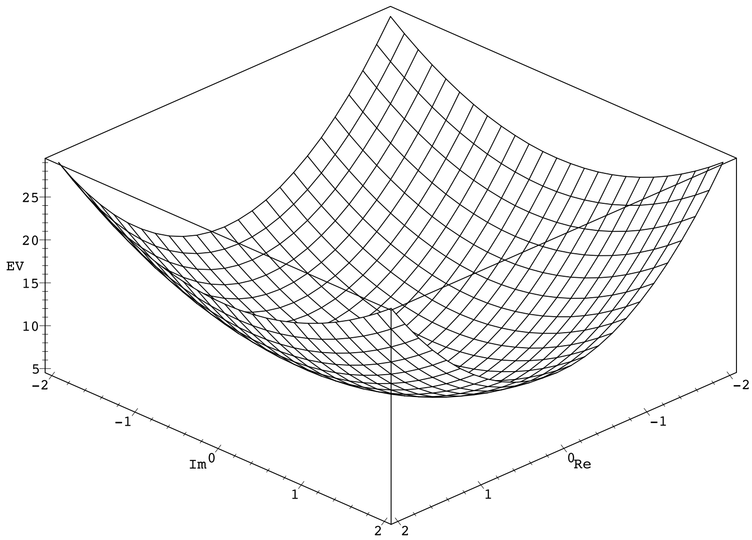

Figure 2. The expectation value $\langle\hat{X}^2\rangle$ in the CSCS, as a function of α for the particular case of $\ell = 1$. In this figure $EV \equiv \langle\hat{X}^2\rangle$, Re $\equiv \mathrm{Re}\,(\alpha)$ and Im $\equiv \mathrm{Im}\,(\alpha)$. The behaviour of $\langle\hat{P}^2\rangle$ is similar.

The introduction of new position and momentum operators establishing a connection between the new and canonical CS is not only a characteristic of the systems with a WH algebra [17, 18].

From (45) we obtain the following expression for $g_-(|\alpha|)$:

$$g_- = g_-(|\alpha|; \ell = 1) = \frac{4|\alpha|\cosh(|\alpha|) - 4\sinh(|\alpha|)}{\sqrt{\pi}|\alpha|^3} \qquad (70)$$

for $\ell = 1$, which is divergent as $|\alpha| \to \infty$. In this case, from (44) and (45) we see that as $\ell = 1$ the asymptotic behaviour of the CSCS is null.

5. Concluding remarks

In this work, the interesting general question as to the possibility of a direct SUSY formulation à la Witten for the SCS possibly associated with the SUSY Hamiltonian systems has been raised and answered affirmatively for the particular case of the CSCS associated with a SUSY isotonic oscillator (harmonic plus a centripetal barrier) system. The existence of a graded algebraic structure akin to the QMSUSY algebra but now governing a system of annihilation operator eigenvalue equations associated with the SUSY isotonic oscillator has been a new output, in which analysis the super Wigner oscillator connection has been an essential input. The scenario of SUSY partner annihilation operators of the SUSY annihilation operator then emerged naturally for this system. The SUSY features of two SUSY isotonic Hamiltonians that were defined in the theory, one with unbroken SUSY and the other with broken SUSY, have been exploited to deduce the explicit expressions for the bosonic and fermionic components of the associated SUSY CSCS, individually satisfying normalization, non-orthogonality and completeness relations and together getting SUSY-intertwined by the charge operators of the new SUSY algebra of annihilation operators only. In [19], coherent states of the isotonic oscillator have been studied in the framework of the generalized quantum conditions.

Although we have mainly treated a SUSY formulation of the SUSY isotonic oscillator CSCS, similar results can be adequately extracted for any physical D-dimensional radial SUSY oscillator system by the Hermitian replacement of $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \to -\mathrm{i}(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{D-1}{2r})$ and of the Wigner parameter $\ell + 1 \to \ell_D + \frac{1}{2}(D-1)$ where $\ell_D(\ell_D = 0, 1, 2, \ldots)$ is the D-dimensional oscillator angular momentum. Also, the present formalism suggests guidelines for SUSY extension to SCS associated with other SUSY systems such as that of the Coulomb problem and the Pöschl–Teller potentials via the Wigner connection as these potentials are intrinsically oscillator-connected [10]. The subject of supersqueezed states is also being developed [20]. In fact, the relaxation of equal dispersion conditions on the Wigner position $\hat{x}$ and momentum $\hat{p}$ will include a description of isotonic canonical supersqueezed states as the minimum product uncertainty states which, as will be published elsewhere, also admit a SUSY formulation analogous to the present one for the SUSY CSCS. Studies on inter-connections of SUSY CSCS with SCS emanating from other definitions such as the supergroup extended displacement operator one [4] will also be pursued.

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References

[1] Witten E 1981 *Nucl. Phys.* B **185** 513
Salomonson P and Van Holten J W 1982 *Nucl. Phys.* B **196** 509
de Crombrugghe M and Rittenberg V 1983 *Ann. Phys., NY* **151** 99
[2] Gendenshteĭn L É and Krive I V 1985 *Sov. Phys.–Usp.* **28** 645
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** L57
Haymaker R W and Rau A R P 1986 *Am. J. Phys.* **54** 928

[3] Aragone C and Zypman F 1986 *J. Phys. A: Math. Gen.* **19** 2267
[4] Fatyaga B W, Kostelecký V A, Nieto M M and Truax D R 1991 *Phys. Rev.* D **43** 1403
[5] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
Perelomov A M 1986 *Generalized Coherent States and their Applications* (Berlin: Springer)
[6] Balantekin A B, Schmitt H A and Barrett B R 1988 *J. Math. Phys.* **29** 1634
See also Balantekin A B, Schmitt H A and Hase P 1989 *J. Math. Phys.* **30** 274
[7] Wigner E P 1950 *Phys. Rev.* **77** 711
Yang L M 1951 *Phys. Rev.* **84** 788
O'Raifeartaigh L and Ryan C 1963 *Proc. R. Ir. Acad.* A **62** 93
Ohnuki Y and Kamefuchi S 1978 *J. Math. Phys.* **19** 67
Ohnuki Y and Watanabe S 1992 *J. Math. Phys.* **33** 3653
[8] Sharma J K, Mehta C L and Sudarshan E C G 1978 *J. Math. Phys.* **19** 2089
Mukunda N, Sudarshan E C G, Sharma J K and Mehta C L 1980 *J. Math. Phys.* **21** 2386
Sharma J K, Mehta C L, Mukunda N and Sudarshan E C G 1981 *J. Math. Phys.* **22** 78
[9] Macfarlane A J 1994 *J. Math. Phys.* **35** 1054
[10] Jayaraman J and de Rodrigues R L 1990 *J. Phys. A: Math. Gen.* **23** 3123
[11] Jayaraman J 1990 Wigner's Super-quantum rule for the spectra of Pöschl–Teller potentials I and II *11th Encontro Nacional de Física de Partículas e Campos (Caxambu-MG, Brazil, 1990)* (in Portuguese)
Jayaraman J 1990 The Wigner–Heisenberg algebra as an operator technique for spectral resolutions of diverse shape-invariant potentials *Paper Presented at 8th Encontro de Físicos do Nordeste (Maceió-AL, Brazil, 1990)* (in Portuguese)
[12] Beckers J and Debergh N 1993 *Phys. Lett.* A **178** 43
[13] Gendenshteĭn L É 1983 *Pis'ma Zh. Eksp. Teor. Fiz.* **38** 299 (Engl. transl. 1983 *JETP Lett.* **38** 356)
Dutt R, Khare A and Sukhatme U P 1988 *Am. J. Phys.* **56** 163
Cooper F, Ginocchio J N and Khare A 1987 *Phys. Rev.* D **36** 2458
Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[14] Abramowitz M 1965 *Handbook of Mathematical Functions* (New York: Dover) pp 374–6
[15] Basu D 1992 *J. Math. Phys.* **33** 114
[16] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series, and Products* 4th edn (New York: Academic) p 684
[17] Nieto M M 1984 *Phys. Rev.* D **30** 770
Nieto M M and Simmons Jr L M 1979 *Phys. Rev.* D **20** 1332
Gutschick V P, Nieto M M and Simmons L M Jr 1980 *Phys. Lett.* A **76** 15
[18] Fernández D J, Nieto L M and Rosas-Ortiz O J 1995 *J. Phys. A: Math. Gen.* **28** 2693
Rosas-Ortiz O J 1996 *J. Phys. A: Math. Gen.* **29** 3281
[19] Bagchi B and Bhaumik D 1998 *Mod. Phys. Lett.* A **13** 811
[20] Kostelecky V A, Nieto M M and Truax D R 1993 *Phys. Rev.* A **48** 1045
Nieto M M Supersqueezed states from squeezed states *Preprint* LA-UR-92-2774
Nieto M M 1994 Coherent states and squeezed states, supercoherent states and supersqueezed states *On Klander's Path: a Field Trip* ed G E Emch, G C Hegerfeldt and L Streit (Singapore: World Scientific) p 147
(Nieto M M Coherent states and squeezed states, supercoherent states and supersqueezed states *Preprint* LA-UR-92-1284)